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Chebyshev Pseudospectral Collocation Method to Unsteady Squeezing Flow of MHD Third grade Nanofluid between Two Disks in a Porous Medium under the Influence of Thermal Radiation

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Abstract: The present paper focuses on the application of Chebyshev pseudospectral collocation method to nonlinear analysis of unsteady squeezing flow and heat transfer of a third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump boundary conditions. The parametric studies from the numerical simulations show that for a suction parameter greater than zero, the radial velocity of the lower disc increases while that of the upper disc decreases because of a corresponding increase in the viscosity of the fluid from the lower squeezing disc to the upper disc. An increasing magnetic field parameter, the radial velocity of the lower disc decreases while that of the upper disc increases. As the third-grade fluid parameter increases, there is a reduction in the fluid viscosity thereby increasing resistance between the fluid molecules. There is a recorded decrease in the fluid temperature profile as the Prandtl number increases due to decrease in the thermal diffusivity of the third-grade fluid. The results in this work can be used to advance the analysis and study of the behaviors of third grade nanofluid flow and heat transfer processes such as found in coal slurries, polymer solutions, textiles, ceramics, catalytic reactors, and oil recovery applications.

Keywords: Third grade nanofluid; Squeezing flow; Magnetohydrodynamic; Thermal radiation; Temperature jump boundary conditions, Chebyshev pseudospectral collocation method

1. Introduction

Over the last decades, the research area that considered how two parallel discs or surfaces can be properly studied to improve their efficiencies as applied to squeezing flow between them has really gained interest and outstanding attention because of its tremendous applications in vital fields like in engineering, industrial and in biological applications. These interesting applications of squeezing flow between two parallel surfaces as stated above have inspired various researchers to critically study the problem internally by considering the flow field together with the squeezing agents and externally by considering other factors at the boundary of the system under analysis [1-10].

In other to provide an extension to the works of the above researchers as required, studies have been performed on squeezable fluids of different nano-particles sizes, different concentration, different stretching effect and different phases by taking slip into account [11-22]. The obtained model for the process was documented and proper parametric studies were performed to understand the process in general [23-34]. Past research works have presented magnetohydrodynamic fluid flow considering porous medium, nanofluid and some other factors capable of reshaping the squeezing process from idealization into actual [35-76]. In recent times, the flow and heat transfer characteristics of third grade nanofluid in pipes and in porous channel have been analyzed [77-89].

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Different schemes have also been employed to efficiently solve the resulting ordinary differential equations associated with squeezing fluid flow between two parallel surfaces as presented by [90-96].

2. Problem Formulation

Given a solid rectangular fin that is subjected to thermal convective and radiation effects as shown in figure 1, the fin is acted upon by identical magnetic field placed perpendicularly (y-direction) to fin surface. The fin material is taken to be homogenous and isotropic with constant physical and thermal properties except thermal conductivity.

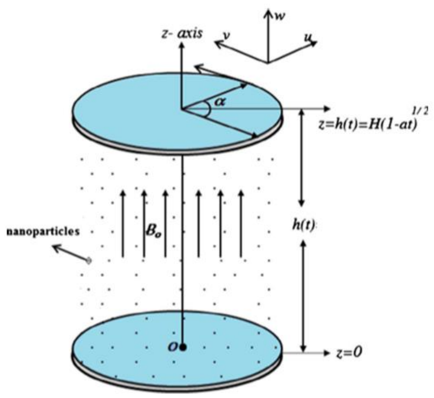

Fig. 1 – The squeezing flow of Third grade nanofluid between parallel circular plate

The Cauchy stress tensor, $\boldsymbol{\tau}$ for an incompressible homogeneous thermodynamically compatible third grade fluid is given by Eqn. (A1), where $\boldsymbol{\tau}$ is the stress tensor, ρ is the pressure, $\boldsymbol{I}$ is the identity tensor, μ is the dynamic viscosity, α_1, α_2, β_1, β_2, and β_2 are the material constant and A_1, A_2 and A_3 being the first, second and third Rivilin Erickson tensor which can be derived from the Eqns. (A2 and A3). $\boldsymbol{V}$ in the equations denotes velocity field, grad is the operator gradient and d/dt is the material time derivative.

According to Fosdick Rajagopal [69] motion of the fluid must be in accordance with the thermodynamic model (Clausius Duhem inequality must be satisfied). When the motions of the fluid are thermodynamically compatible, the Clausius-Duhem inequality and the assumption that the Helmholtz free energy is minimum when the fluid is locally at rest (stable) require that,

$$\mu \geq 0, \alpha_1 \geq 0, \beta_1 = \beta_2 = 0,$$
$$\beta_3 \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},$$
$$\beta_1 = \beta_2 = 0, \quad \beta \geq 0$$

Since $\beta_3 > 0$, the stress tensor can predict shear thickening as well as the normal stress. Velocity and temperature fields are,

$$\mathbf{V} = \left(u(t,r,z), 0, w(t,r,z)\right) \text{ and } \mathbf{T} = T(t,r,z)$$

Here, u and w are the radial and axial components of velocity. From the Eqn. A1 and the relation between radial and axial components of velocity, the algebraic forms of the conservation equations can be developed as Eqns. (A4-A6). The tensor notations given an in Eqns. (A7-A10). Substitution of Eqns. (A7-A8) into the momentum equations in Eqn. (A4) and (A5) and expansion of the resulting equations, one arrives at Eqns. (A11) and (A12). And the energy equation is given as in Eqn. (A13).

The related boundary conditions are given as,

$$u(r,z,t) = U_w = \frac{ar}{2(1-ct)}$$
$$w(r,z,t) = -w_0$$
$$-k\frac{\partial T}{\partial z} = h_1\left(T_f - T\right) \text{ at } z = 0,$$
$$u(r,z,t) = 0$$
$$w(r,z,t) = \frac{\partial h}{\partial t} = -\frac{c}{2}\sqrt{\frac{v}{a(1-ct)}}$$
$$-k\frac{\partial T}{\partial z} = h_2\left(T - T_h\right) \text{ at } z = h(\mathrm{t}),$$

where $w_0 > 0$ represents suction. The various physical and thermal properties are given as,

$$\rho_{nf} = \rho_f(1-\phi) + \rho_s\phi$$

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}$$

$$\left(\rho c_p\right)_{nf} = \left(\rho c_p\right)_f (1-\phi) + \left(\rho c_p\right)_s \phi$$

$$(\rho\beta)_{nf} = (\rho\beta)_f (1-\phi) + (\rho\beta)_s \phi$$

$$\sigma_{nf} = \sigma_f\left[1 + \frac{3\left\{\frac{\sigma_s}{\sigma_f} - 1\right\}\phi}{\left\{\frac{\sigma_s}{\sigma_f} + 2\right\}\phi - \left\{\frac{\sigma_s}{\sigma_f} - 1\right\}\phi}\right],$$

$$k_{nf} = k_f\left[\frac{k_s + 2k_f - 2\phi\left(k_f - k_s\right)}{k_s + 2k_f + \phi\left(k_f - k_s\right)}\right]$$

Table 1 – Physical and thermal properties of the base fluid

Base fluid	ρ (kg/m^3)	c_p (J/kgK)	k (W/mK)
Pure water	997.1	4179	0.613
Ethylene Glycol	1115	2430	0.253
Engine oil	884	1910	0.144
Kerosene	783	2021	0.145

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Table 2 – Physical and thermal properties of nanoparticles

Nanoparticles	ρ (kg/m^3)	c_p (J/kgK)	k (W/mK)
Copper (Cu)	8933	385	401
Al. oxide (Al_2O_3)	3970	765	40
SWCNTs	2600	42.5	6600
Silver (Ag)	10500	253	429
Titanium dioxide (TiO_2)	4250	686.2	8.954
Copper Oxide (CuO)	783	540	18

Table 1 and 2 present the physical and thermal properties of the base fluid and the nanoparticles, respectively. SWCNTs mean single-walled carbon nanotubes. We consider the following transformations:

$$\eta=\frac{z}{h(t)},\ u=U_w f'(\eta)$$
$$w=-\sqrt{\frac{av}{1-ct}}f(\eta),\ \theta=\frac{T-T_h}{T_f-T_h}$$

Identically, it is established that the continuity equation is satisfied. Omitting the pressure terms from the momentum equations in Eqns. (A11) and (A12) and using the above transformations, one obtains,

$$\begin{aligned}&f^{iv}+ff'''-\frac{Sq}{2}\left(3f''-\eta f'''\right)-\left(M^2+\frac{1}{Da}\right)f''\\&+\alpha\left(-2f'f'''-ff^{iv}-ff^{v}+\frac{Sq}{2}\left(5f^{iv}-\eta f^{v}\right)\right)\\&-\gamma\left(2f'f'''+ff^{iv}\right)+\beta\begin{pmatrix}7f''^3+24ff'f'''+3f'^2f^{iv}\\+\mathrm{Re}\left(3f'f'''^2+\frac{3}{2}f''^2f^{iv}\right)\end{pmatrix}=0,\end{aligned}\tag{1}$$

and,

$$\begin{aligned}&(1+Rd)\theta''+\Pr\left(\theta'f-\frac{Sq}{2}\eta\theta'\right)\\&+PrEc\left\{\begin{array}{l}M^2f'^2+\frac{6}{\mathrm{Re}}f'^2+f''^2\\+\alpha\begin{pmatrix}\frac{1}{\mathrm{Re}}\left(-6f'^3-6ff'f''\right)\\-ff''^2-ff'f'''+\frac{Sq}{2}\left(3f''^2+\eta f'f'''\right)\\+\frac{Sq}{\mathrm{Re}}\left(6f'^2+3\eta ff''\right)\end{pmatrix}\\-3\gamma\left(\frac{2f'^3}{\mathrm{Re}}+\frac{ff''^2}{2}\right)+\beta\begin{pmatrix}\frac{18}{\mathrm{Re}}f'^4\\+6f'^2f''^2+\frac{\mathrm{Re}}{2}f''^4\end{pmatrix}\end{array}\right\}=0\end{aligned}\tag{2}$$

The corresponding boundary conditions are,

$$f(0)=A,\ f(1)=S_q/2,\quad f'(0)=1,\quad f'(1)=0,\tag{3}$$

$$\theta'(0)=\Upsilon_1\left(\theta(0)-1\right),\ \theta'(1)=-\Upsilon_2\theta(1)\tag{4}$$

where,

$$\alpha=\frac{\alpha_1 a}{\mu(1-ct)},\quad \beta=\frac{2\beta_3a^2}{\mu(1-ct)^2},\quad \gamma=\frac{\alpha_2 a}{\mu(1-ct)}$$
$$\mathrm{Re}=\frac{ar^2}{2v(1-ct)},\quad \Pr=\frac{\mu C_p}{K},\quad Ec=\frac{U_w^2}{C_p\left(T_f-T_h\right)}$$
$$M^2=\frac{\sigma B_0^2}{\rho a},\quad Sq=\frac{c}{a},\quad A=\sqrt{\frac{(1-ct)}{av}}w_0$$
$$Rd=\frac{16\sigma^*T_h^3}{3k_1^*K},\quad \Upsilon_1=\frac{h_1h(\mathrm{t})}{K},\quad \Upsilon_2=\frac{h_2h(\mathrm{t})}{K},$$

3. Numerical Solution using Chebyshev Pseudospectral Collocation Method

Eqns. (1) and (2) are systems of coupled non-linear ordinary differential equations which are to be solved alongside the boundary conditions in Eqns. (3) and (4). To solve the nonlinear equations, we applied Chebyshev pseudospectral collocation method. Following the method procedure, the η direction domain is approximated by [*0*, *L*], where *L* is the edge of the boundary layer. Using the algebraic mapping $\eta=(\gamma+1)L/2$, the physical region can be mapped out into Chebyshev collocation method computation domain of [*-1*, *1*]. The unknown variables in Eqns. (1) and (2) are approximated using the following truncated series of polynomials:

$$f(\lambda)=\sum_{k=0}^{N}\hat{f}_kT_k(\lambda),\quad \theta(\gamma)=\sum_{k=0}^{N}\hat{\theta}_kT_k(\lambda)\tag{3}$$

where the values of $\hat{f}_k$ and $\hat{\theta}_k$ are the coefficient of the series to be determined, $\hat{T}_k$ is the Chebyshev polynomial of degree k which is defined on the interval $\lambda \in [-1, 1]$.

$$T_k(\lambda)=cos\left(kcos^{-1}\lambda\right),\quad k=0,1,...\tag{4}$$

The application of pseudospectral collocation method requires that the spatial discretization of the dimensionless models should be carried out through the utilization of Chebyshev-Gauss-Lobato collocation points. Using the Gauss-Lobato collocation points to define the Chebyshev nodes in [-1,1] as,

$$\lambda_j=cos\frac{\pi j}{N},\quad -1\le\gamma\le1,\ j=0,1,...,N\tag{5}$$

The collocation points are mapped out in computation domain of [*-1*, *1*]. To transform any arbitrary (*X*, *[0*, *1*]) interval into standard interval (ψ, [*-1*, *1*]), a transformation using algebraic mapping $X=(\psi+1)/2$ is adopted. The derivatives of the functions f and θ at the collocation points are represented as,

$$\frac{df}{d\eta}=\frac{2}{L}\sum_{k=0}^{N}\mathbf{D}_{jk}\hat{f}_k,\quad \frac{d\theta}{d\eta}=\frac{2}{L}\sum_{k=0}^{N}\mathbf{D}_{jk}\hat{\theta}_k\tag{6}$$

where **D** is the Chebyshev spectral method differentiation matrix. Higher order derivatives are computed as multiple powers of **D**, i.e.,

$$\frac{d^i f}{d\eta^i} = \left(\frac{2}{L}\right)^i \sum_{k=0}^{N} \mathbf{D}^i_{jk} \hat{f}_k, \quad \frac{d^i \theta}{d\eta^i} = \left(\frac{2}{L}\right)^i \sum_{k=0}^{N} \mathbf{D}^i_{jk} \hat{\theta}_k \qquad (7)$$

where i is the order of the derivatives.

On substituting Eqns. (6) - (6) into the governing equations and the boundary conditions in Eqns. (3) and (4), respectively, the following equations are obtained.

$$\begin{aligned}
&\bar{\mathbf{D}}^4\mathbf{F} + \mathbf{F}\left(\bar{\mathbf{D}}^3\mathbf{F}\right) - \frac{Sq}{2}\left(3\bar{\mathbf{D}}^2\mathbf{F} - \eta\bar{\mathbf{D}}^3\mathbf{F}\right) - \left(M^2 + \frac{1}{Da}\right)\bar{\mathbf{D}}^2\mathbf{F} \\
&+\alpha\begin{pmatrix} -2\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right) - \left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^4\mathbf{F}\right) - \mathbf{F}\left(\bar{\mathbf{D}}^5\mathbf{F}\right) \\ + \frac{Sq}{2}\left(5\bar{\mathbf{D}}^4\mathbf{F} - \eta\bar{\mathbf{D}}^5\mathbf{F}\right) \end{pmatrix} \\
&-\gamma\left(2\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right) + \left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^4\mathbf{F}\right)\right) \\
&+\beta\begin{pmatrix} 7\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^3 + 24\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right) + 3\left(\bar{\mathbf{D}}\mathbf{F}\right)^2\left(\bar{\mathbf{D}}^4\mathbf{F}\right) \\ + \mathrm{Re}\left(3\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right)^2 + \frac{3}{2}\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2\left(\bar{\mathbf{D}}^4\mathbf{F}\right)\right) \end{pmatrix} = 0,
\end{aligned} \qquad (8)$$

$$\begin{aligned}
&(1+Rd)\bar{\mathbf{D}}^2\Theta + Pr\left(\mathbf{F}\bar{\mathbf{D}}\Theta - \frac{Sq}{2}\eta\bar{\mathbf{D}}\Theta\right) \\
&+PrEc\left\{\begin{array}{l} M^2\left(\bar{\mathbf{D}}\mathbf{F}\right)^2 + \frac{6}{\mathrm{Re}}\left(\bar{\mathbf{D}}\mathbf{F}\right)^2 + \left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 \\
+\alpha\begin{pmatrix} \frac{1}{\mathrm{Re}}\left(-6\left(\bar{\mathbf{D}}\mathbf{F}\right)^3 - 6\mathbf{F}\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\right) \\ -\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 - \mathbf{F}\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right) \\ + \frac{Sq}{2}\left(3\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 + \eta\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right)\right) \\ + \frac{Sq}{\mathrm{Re}}\left(6\left(\bar{\mathbf{D}}\mathbf{F}\right)^2 + 3\eta\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\right) \end{pmatrix} \\
-3\gamma\left(\frac{2\left(\bar{\mathbf{D}}\mathbf{F}\right)^3}{\mathrm{Re}} + \frac{\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2}{2}\right) \\
+\beta\left(\frac{18}{\mathrm{Re}}\left(\bar{\mathbf{D}}\mathbf{F}\right)^4 + 6\left(\bar{\mathbf{D}}\mathbf{F}\right)^2\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 + \frac{\mathrm{Re}}{2}\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^4\right)
\end{array}\right\} = 0
\end{aligned} \qquad (9)$$

where,

$$\bar{\mathbf{D}} = (2/L)\mathbf{D},\ \mathbf{F} = \left\{\hat{f}_0, \hat{f}_1, ..., \hat{f}_N\right\}^T, \Theta = \left\{\hat{\theta}_0, \hat{\theta}_1, ..., \hat{\theta}_N\right\}^T$$

and the superscript T denotes transpose. And the boundary conditions become,

$$\hat{f}_0 = A, \quad \sum_{k=0}^{N} \hat{f}_k = \frac{S_q}{2}, \quad \hat{\mathbf{D}}\hat{f}_0 = 1, \quad \sum_{k=0}^{N} \hat{\mathbf{D}}_{Nk}\hat{f}_k = 0 \qquad (10)$$

$$\hat{\mathbf{D}}\Theta_0 = \Upsilon_1\left(\Theta_0 - 1\right), \quad \sum_{k=0}^{N} \hat{\mathbf{D}}_{Nk}\Theta_k = -\Upsilon_2\sum_{k=0}^{N}\Theta_k \qquad (11)$$

Although, the algorithm for the implementation of the Chebyshev pseudospectral collocation method is given in the next section, it should be stated that Eqn. (8) can be solved separately subject to the boundary conditions in Eqn. (10) using a MATLAB nonlinear equation solver *fzero* which is based on the quasi-Newton method. Also, it should be noted that Eqn. (1) or Eqn. (8) are the third order differential equation. There are four boundary conditions for this equation as depicted in Eqn. (3) or Eqn. (10). One of the ways to make the governing equation satisfy all the boundary conditions is to differentiate the governing equation in Eqn. (1) or Eqn. (10) to fourth order. Once, the solution for $\boldsymbol{F}$ is obtained, it is then substituted into equation Eqn. (9) to get,

$$\mathbf{M}\Theta = \mathbf{K}, \quad \text{then} \quad \Theta = \mathbf{M}^{-1}\mathbf{K}, \qquad (12)$$

where,

$$\begin{aligned}
\mathbf{M} &= (1+Rd)\bar{\mathbf{D}}^2 + Pr\left(\mathbf{F}\bar{\mathbf{D}} - \frac{Sq}{2}\eta\bar{\mathbf{D}}\right) \\
&+PrEc\left\{\begin{array}{l} M^2\left(\bar{\mathbf{D}}\mathbf{F}\right)^2 + \frac{6}{\mathrm{Re}}\left(\bar{\mathbf{D}}\mathbf{F}\right)^2 + \left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 \\
+\alpha\begin{pmatrix} \frac{1}{\mathrm{Re}}\left(-6\left(\bar{\mathbf{D}}\mathbf{F}\right)^3 - 6\mathbf{F}\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\right) \\ -\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 - \mathbf{F}\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right) \\ + \frac{Sq}{2}\left(3\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 + \eta\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\left(\bar{\mathbf{D}}^3\mathbf{F}\right)\right) \\ + \frac{Sq}{\mathrm{Re}}\left(6\left(\bar{\mathbf{D}}\mathbf{F}\right)^2 + 3\eta\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)\right) \end{pmatrix} \\
-3\gamma\left(\frac{2\left(\bar{\mathbf{D}}\mathbf{F}\right)^3}{\mathrm{Re}} + \frac{\left(\bar{\mathbf{D}}\mathbf{F}\right)\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2}{2}\right) \\
+\beta\left(\frac{18}{\mathrm{Re}}\left(\bar{\mathbf{D}}\mathbf{F}\right)^4 + 6\left(\bar{\mathbf{D}}\mathbf{F}\right)^2\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^2 + \frac{\mathrm{Re}}{2}\left(\bar{\mathbf{D}}^2\mathbf{F}\right)^4\right)
\end{array}\right\}
\end{aligned}$$

$\boldsymbol{K}$ = [0 0 … 0 1] ^{T} and $\boldsymbol{I}$ is the identity matrix of size $N+1$. The boundary conditions are imposed by setting the first and last row of $\boldsymbol{M}$ to be [1 0 … 0 0] and [0 0 … 0 1].

3.1 Numerical Solution using Chebyshev Pseudospectral Collocation Method

Using the following routine, the Chebyshev pseudospectral collocation method can be implemented as follows:

Step 1: Input the number of collocations points and compute the coordinate values of the nodes for the matrices of the first-order, second order and third-order derivatives.

Step 2: Set the dimensionless velocity and temperature at the initial values in all directions except at the boundaries.

Step 3: Use Eqns. (8) and (9), assemble the matrices**.**

Step 4: Imposed the boundary conditions in Eqn. (10) in the algebraic systems of equations and solve the equation directly.

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Step 5: Using the convergence criteria, terminate the iteration procedure, otherwise, return to Step 3. The convergence criteria:

$$\sum \left| \phi_i^p - \phi^{p-1} \right| \leq 10^{-12}$$

Another set of important considerations in the analysis of fluid flow and heat transfer are skin friction coefficient and Nusselt number.

The local skin friction coefficient at lower disk,

$$C_f = \frac{\tau_w |_{x=0}}{\frac{1}{2}\rho U_w^2}$$

While the local Nusselt number at the disk,

$$Nu = \frac{h(\mathrm{t}) q_w}{K\left(T_f - T_h\right)} |_{z=0}$$

where wall heat flux is defined as,

$$q_w |_{z=0} = -K \frac{\partial T}{\partial z} |_{z=0} + q_r |_{z=0}$$

Using the similarity variables in Eq. (20) and the dimensionless parameters in Eq. (25), one obtains the dimensionless forms of the local skin friction coefficient and Nusselt number as,

$$C_f = \sqrt{2}\,\mathrm{Re}^{-0.5} \left(\begin{aligned} &2f''(0) + \alpha\left(3Sqf''(0) - 2Af'''(0) + 2f''(0)\right) \\ &-2\gamma f''(0) + \beta\left(\frac{\mathrm{Re}}{4} f'''^{3}(0) + 6f''(0)\right) \end{aligned} \right)$$

$$Nu = \left(1 + Rd\right)\theta'(0)$$

4. Results and Discussion

The results of the MATLAB simulation of the developed modeled Chebyshev pseudospectral collocation method as presented in the previous section are illustrated in this section. Parametric studies are carried out and the influences of various parameters on the flow and heat transfer processes are established as shown in figures 2-18.

Figures 2 to 5 depict the influence of fluid flow parameters on dimensionless velocity profile. It is evident that an increase in the fluid flow parameters of the squeezing flow causes a corresponding increase in the fluid flow velocity. This is because the fluid flow parameters in question vary inversely with the viscosity of the fluid being squeezed. As these parameters increase, the viscosity of the fluid decreases and consequently increases the velocity of the fluid as the molecules of the squeezing flow are free to move with less restriction. These parameters can be used as a monitoring agent as they directly affect the viscosity of the third grade squeezing fluid.

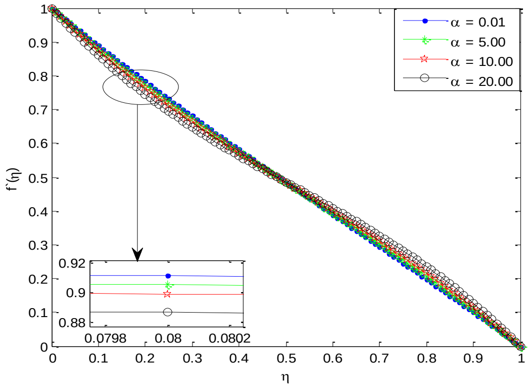

Fig. 2 – Influence of α on dimensionless velocity profile

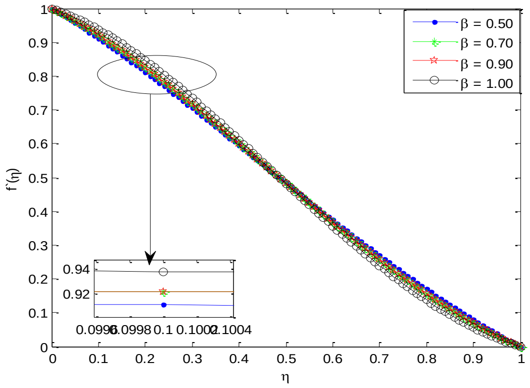

Fig. 3 – Influence of β on dimensionless velocity profile

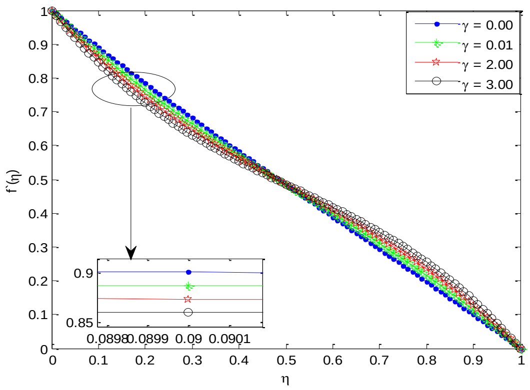

Fig. 4 – Influence of γ on dimensionless velocity profile

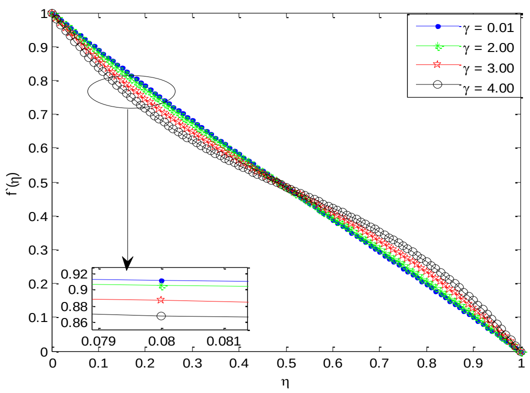

Fig. 5 – Influence of γ on dimensionless velocity profile

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Figures 6 and 7 depict the Influence of suction and squeezing parameters on dimensionless velocity profile. Figure 6 shows how the suction parameter affects the velocity of the squeezing discs for an increasing value of the fluid parameters. It is obvious that for a suction parameter greater than zero, the radial velocity of the lower disc increases while that of the upper disc decreases as a result of a corresponding third-grade the viscosity of the third grade fluid from the lower squeezing disc to the upper disc. Fig. 7 depicts the impact of the squeezing parameter on the fluid flow between the parallel discs. The figure shows that an increase in the squeezing parameter causes a corresponding increase in the squeezing rate. This is because as the squeezing parameter increases, the radial velocity of the squeezing discs increases there by generating a driving compressive rotary force on the fluid flowing between the two parallel discs.

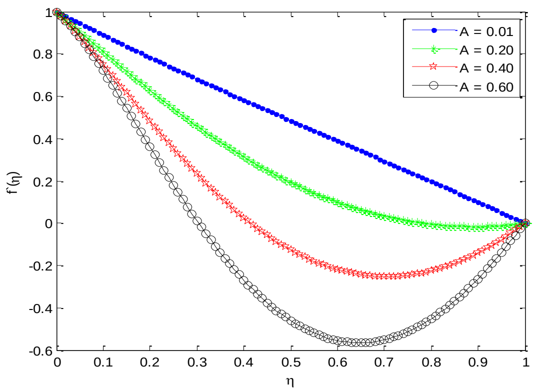

Fig. 6 – Influence of suction term on dimensionless velocity profile

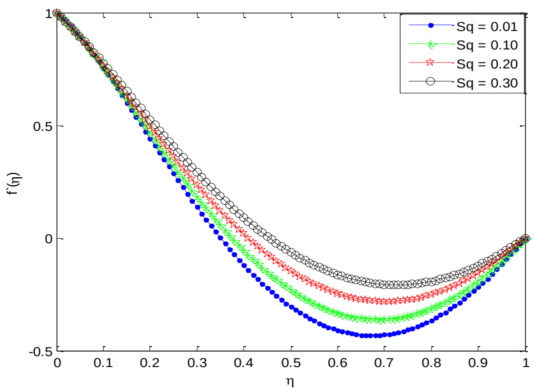

Fig. 7 – Influence of squeezing term on dimensionless velocity profile

Figures 8 and 9 depict the influence of Hartman number and Reynold's number on dimensionless velocity profile. Figure 8 shows how the Hartman number affects the velocity of the squeezing discs for an increasing value of the fluid parameters. It is obvious that for a Hartman number greater than zero or for an increasing Hartman number, the radial velocity of the lower disc decreases while that of the upper disc increases as a result of a corresponding increase in the viscosity of the third grade fluid from the upper disc to the lower squeezing disc.

As the Hartman number becomes large, it automatically raises the magnetic field because of a corresponding increase in the Lorentz force, hence decreases the flow velocity while increase in Reynold's number increases the velocity of the fluid as shown in figure 9.

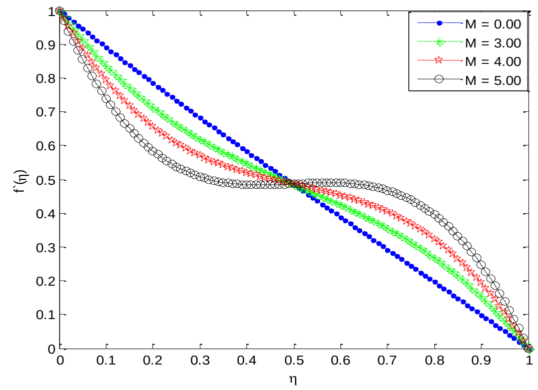

Fig. 8 – Influence of Hartman number on dimensionless velocity profile.

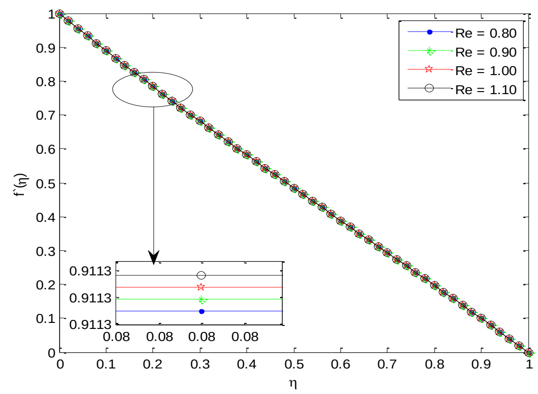

Fig. 9 – Influence of Reynold's number on dimensionless velocity profile.

Figures 10 and 11 depict the influence of the third grade fluid parameter and Reynold's number on dimensionless temperature profile. It has been ascertained that an increase in the third grade fluid parameter causes reduction in the fluid viscosity thereby increasing resistance between the fluid molecules. However, as the Reynold's number associated with the third grade fluid increases, a decreasing effect is noticed in the dimensionless temperature profile. This is because there is a reduction in the convective capability of a high velocity fluid as compared to that with a moderate velocity.

Figures 12 and 13 depict the influence of Prandtl and Eckert number on dimensionless temperature profile. Figure 12 depicts a decrease in temperature profile as the Prandtl number increases. This is because an increase in the Prandtl number reduces thermal diffusivity thereby reducing the temperature profile. In figure 13, Eckert number is observed to have a linear increasing property on the dimensionless temperature profile. This is because of the increase in the total kinetic energy of the fluid which correspondingly elevates the fluid temperature.

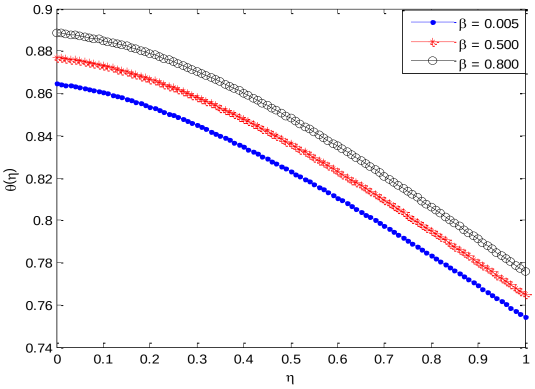

Fig. 10 – Influence of the third grade fluid parameter

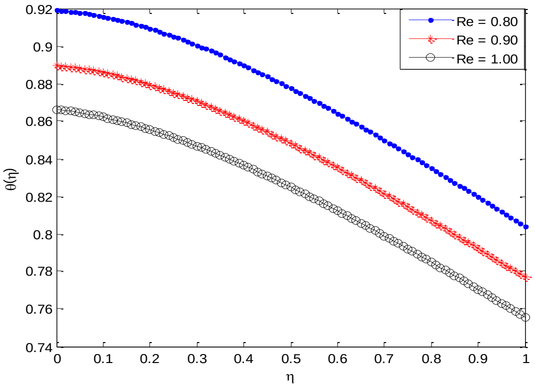

Fig. 11 – Influence of Reynold's number

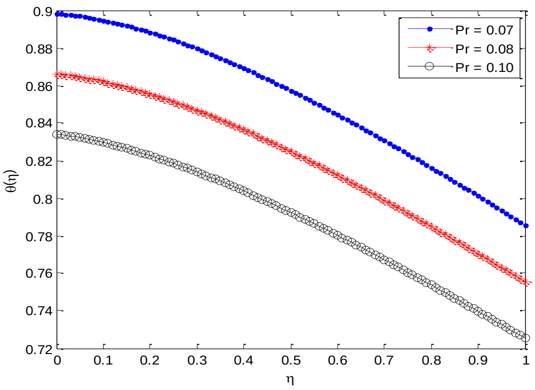

Fig. 12 – Influence of Prandtl number

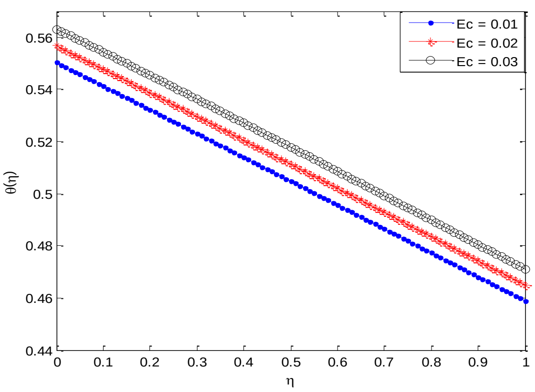

Fig. 13 – Influence of Eckert number on dimensionless

Figures 14 and 15 depict the influence of Hartman number and Squeezing parameter on dimensionless temperature profile. As the Hartman number becomes large, it automatically raises the magnetic field because of an increase in the Lorentz force. This makes the temperature profile increase as the Hartman number increases. Considering figure 15, a rapid increase in the dimensionless temperature profile is noticed for a large value of squeezing parameter because of a driving compressive rotary force which generates a noticeable heating effect thereby increasing the temperature profile.

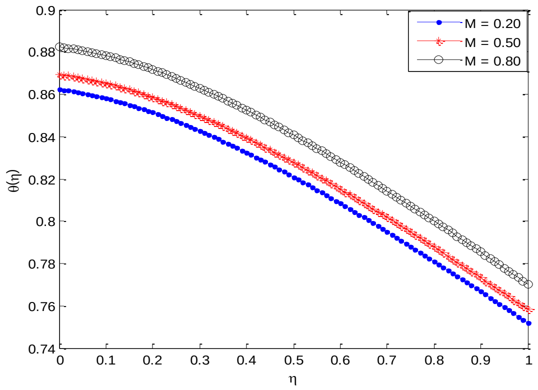

Fig. 14 – Influence of Hartman number on dimensionless temperature profile

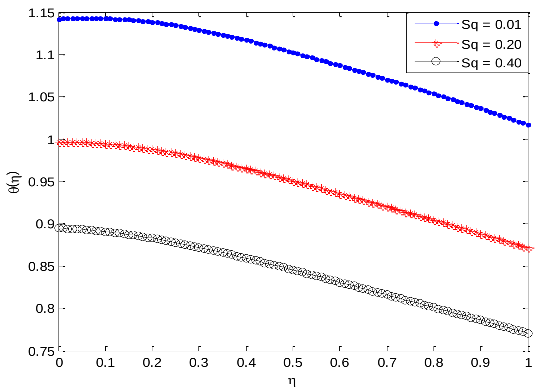

Fig. 15 – Influence of squeezing parameter on dimensionless temperature profile

Figures 16 to 18 depict the influence of the thermal Biot number and Radiation term on dimensionless temperature profile. In figures 16 and 17, the two thermal Biot number have opposing effect on the dimensionless temperature profile, but the cooling effect generated by the first Biot number is more that the temperature rising effect obtained from the second even for the same range of values. As a result, these parameters can serve as a control for temperature monitoring. However, in figure 18, as the radiation parameter increases, the dimensionless temperature profile increases. This is because an increase in the radiation property causes a reduction in the absorptivity and consequently increases the rate of heat transfer to the third grade fluid.

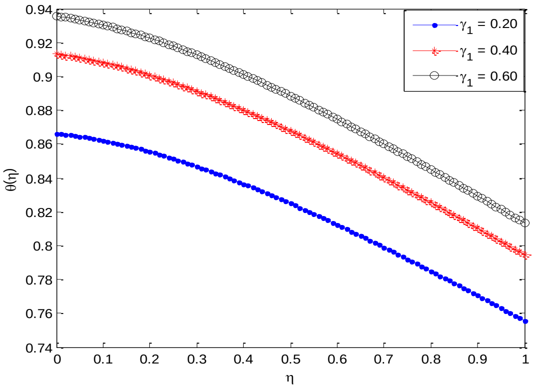

Fig. 16 – Influence of the first thermal Biot number on dimensionless temperature profile

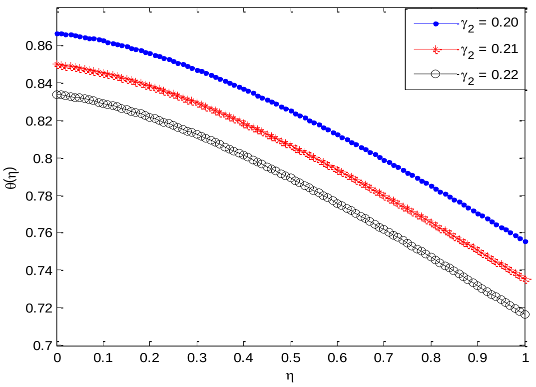

Fig. 17 – Influence of the second thermal Biot number on dimensionless temperature profile

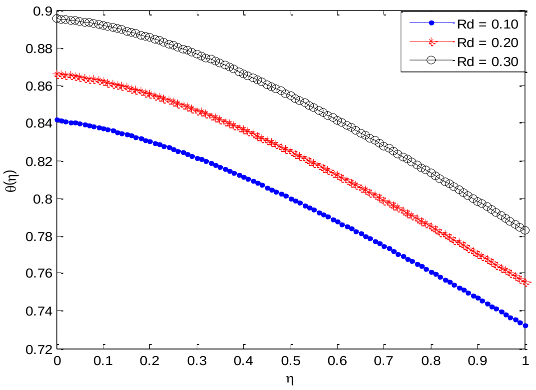

Fig. 18 – Influence of Radiation term on dimensionless temperature profile

5. Conclusion

In this study, nonlinear analysis of unsteady squeezing flow and heat transfer of a third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump boundary conditions was investigated numerically using Chebyshev pseudospectral collocation method. Also, the influences of various flow and heat transfer parameters were investigated. Important significance of study includes the study of flow and heat transfer of third grade fluid as applied in energy conservation, coal slurries, polymer solutions, textiles, ceramics, catalytic reactors, oil recovery applications, friction reduction and micro mixing biological samples.

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Nomenclature

A	Suction/injection parameter
B	Magnetic field strength, Nm/A
c_p	Specific heat capacity at constant pressure, J/kgK
c_w	Specific heat capacity of the nanofluid, J/kgK
k_{nf}	Thermal conductivity of the nanofluid, W/mK
K_p	Permeability of the porous medium, m^2
M	Hartmann number or magnetic field parameter
P	Pressure, N/m^2
Pr	Prandtl's number
r	Radius of the disk, m
h	Height of the flow, m
Sq	Squeeze number
μ_f	Dynamic viscosity of the base fluid, Ns/m^2
μ_{nf}	Dynamic viscosity of the nanofluid, Ns/m^2
ρ_{nf}	Nanofluid density,kg/m^3
α_{nf}	Thermal diffusivity of the nanofluid, m^2/s
$(\rho c_p)_{nf}$	Heat capacity of the nanofluid, Jm^3/K
ρ_f	Density of the basefluid, kg/m^3
θ	**d**imensionless temperature
w	**a**xial velocity component, m/s
u	**r**adial velocity component, m/s
η	Similarity variable
γ	**D**imensionless temperature jump parameter

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Appendices

$$\boldsymbol{\tau} = p\mathbf{I} + \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A_1^2} + \beta_1\mathbf{A_3} + \beta_2\left(\mathbf{A_1A_2} + \mathbf{A_2A_1}\right) + \beta_3\left(\mathbf{trA_1^2}\right)\mathbf{A_1} \tag{A1}$$

$$\mathbf{A_1} = \left(grad\ \mathbf{V}\right) + \left(grad\ \mathbf{V}\right)^T \tag{A2}$$

$$\mathbf{A_n} = \frac{d\mathbf{A_{n-1}}}{dt} + \mathbf{A_{n-1}}\left(grad\ \mathbf{V}\right) + \left(grad\ \mathbf{V}\right)^T \mathbf{A_{n-1}}, \quad n \geq 1 \tag{A3}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{A4}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} - \frac{\sigma B_0^2 u}{1-ct} - \frac{\mu}{K}u \tag{A5}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) + \frac{\partial \tau_{rz}}{\partial z} - \frac{\mu}{K}w \tag{A6}$$

$$\begin{aligned}\tau_{rr} = &-p + 2\mu\frac{\partial u}{\partial r} + 2\alpha_1\left[u\frac{\partial^2 u}{\partial r^2} + w\frac{\partial^2 u}{\partial r\partial z} + 2\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right] \\ &+ \alpha_2\left[4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2\right] + 4\beta_3\frac{\partial u}{\partial r}\left[2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + \left\{2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}\right]\end{aligned} \tag{A7}$$

$$\begin{aligned}\tau_{\theta\theta} = &-p + 2\mu\frac{u}{r} + 2\alpha_1\left[\frac{u}{r}\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial z} + \frac{u^2}{r^2}\right] + 4\alpha_2\frac{u^2}{r^2} \\ &+ 4\beta_3\frac{u}{r}\left[2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + \left\{2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right\}\right]\end{aligned} \tag{A8}$$

$$\begin{aligned}\tau_{zz} = &-p + 2\mu\frac{\partial w}{\partial z} + 2\alpha_1\left[u\frac{\partial^2 w}{\partial r\partial z} + w\frac{\partial^2 w}{\partial z^2} + 2\left(\frac{\partial w}{\partial r}\right)^2 + \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right] \\ &+ \alpha_2\left[4\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2\right] + 4\beta_3\left(\frac{\partial w}{\partial z}\right)\left[\begin{array}{l}2\dfrac{u^2}{r^2} + \left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r}\right)^2 \\ + \left\{2\left(\dfrac{\partial u}{\partial r}\right)^2 + \left(\dfrac{\partial w}{\partial z}\right)^2\right\}\end{array}\right]\end{aligned} \tag{A9}$$

$$\begin{aligned}\tau_{rz} = &\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + \alpha_1\left[\left(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + \frac{\partial u}{\partial r}\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} + 3\left(\frac{\partial u}{\partial r}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right)\right] \\ &+ 2\alpha_2\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + 2\beta_3\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\left[2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + \left\{2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}\right]\end{aligned} \tag{A10}$$

$$
\begin{aligned}
&\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+\\
&\alpha_1\left[\begin{array}{l}
-2\frac{u^2}{r^3}-\frac{2}{r^2}\frac{\partial u}{\partial t}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+5\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{w}{r}\frac{\partial^2 u}{\partial r\partial z}+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2\\
-2\frac{u}{r^2}\frac{\partial u}{\partial r}+4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 w}{\partial r\partial z^2}+3\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}\\
-2\frac{w}{r^2}\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^2\partial t}+w\frac{\partial^3 u}{\partial z^3}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2w\frac{\partial^3 u}{\partial r^2\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
+10\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial^2 u}{\partial r\partial t}+\frac{\partial^3 w}{\partial r\partial z\partial t}+u\frac{\partial^3 u}{\partial r^2\partial z}+2\frac{\partial^3 u}{\partial r^2\partial t}+2u\frac{\partial^3 u}{\partial r^3}+2\frac{u}{r}\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}
\end{array}\right]\\
&+\alpha_2\left[\begin{array}{l}
-4\frac{u^2}{r^3}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\\
+2\frac{\partial^2 w}{\partial r\partial z}\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}
\end{array}\right]\\
&+\beta_3\left[\begin{array}{l}
-8\frac{u^3}{r^4}+8\frac{u^2}{r^2}\frac{\partial^2 u}{\partial r^2}+\frac{8}{r}\left(\frac{\partial u}{\partial r}\right)^3+24\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\\
+16\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+8\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}-4\frac{u}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+16\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
+4\frac{u^2}{r^2}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial^2 u}{\partial r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}-8\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}\\
+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}+4\frac{u^2}{r^2}\frac{\partial^2 u}{\partial z^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+4\frac{u}{r^2}\left(\frac{\partial u}{\partial z}\right)^2\\
+4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}-8\frac{u}{r^2}\left(\frac{\partial w}{\partial z}\right)^2+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}\\
-8\frac{u^2}{r^3}\frac{\partial u}{\partial r}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}
\end{array}\right]-\frac{\sigma B_0^2 u}{1-ct}-\frac{\mu_{nf}}{K}u
\end{aligned}
\tag{A11}
$$

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$$
\begin{aligned}
&\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+\\
&\alpha_1\left[\begin{array}{l}
-2\frac{u^2}{r^3}-\frac{2}{r^2}\frac{\partial u}{\partial t}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+5\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{w}{r}\frac{\partial^2 u}{\partial r\partial z}+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2\\
-2\frac{u}{r^2}\frac{\partial u}{\partial r}+4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 w}{\partial r\partial z^2}+3\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}\\
-2\frac{w}{r^2}\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^2\partial t}+w\frac{\partial^3 u}{\partial z^3}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2w\frac{\partial^3 u}{\partial r^2\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
+10\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial^2 u}{\partial r\partial t}+\frac{\partial^3 w}{\partial r\partial z\partial t}+u\frac{\partial^3 u}{\partial r^2\partial z}+2\frac{\partial^3 u}{\partial r^2\partial t}+2u\frac{\partial^3 u}{\partial r^3}+2\frac{u}{r}\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}
\end{array}\right]\\
&+\alpha_2\left[\begin{array}{l}
-4\frac{u^2}{r^3}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\\
+2\frac{\partial^2 w}{\partial r\partial z}\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}
\end{array}\right] \qquad \text{(A12)}\\
&+\beta_3\left[\begin{array}{l}
-8\frac{u^3}{r^4}+8\frac{u^2}{r^2}\frac{\partial^2 u}{\partial r^2}+\frac{8}{r}\left(\frac{\partial u}{\partial r}\right)^3+24\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\\
+16\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+8\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}-4\frac{u}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+16\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
+4\frac{u^2}{r^2}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial^2 u}{\partial r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}-8\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}\\
+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}+4\frac{u^2}{r^2}\frac{\partial^2 u}{\partial z^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+4\frac{u}{r^2}\left(\frac{\partial u}{\partial z}\right)^2\\
+4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}-8\frac{u}{r^2}\left(\frac{\partial w}{\partial z}\right)^2+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}\\
-8\frac{u^2}{r^3}\frac{\partial u}{\partial r}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}
\end{array}\right]-\frac{\sigma B_0^2 u}{1-ct}-\frac{\mu_{nf}}{K}u
\end{aligned}
$$

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$$
\begin{aligned}
&\left(\rho c_p\right)_{nf}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\mu\left[2\left(\frac{\partial u}{\partial r}\right)^2+2\frac{u^2}{r^2}+\left(\frac{\partial u}{\partial z}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial r}\right)^2\right]\\
&+\alpha_1\left[\begin{aligned}
&2\frac{u^3}{r^3}+4\left(\frac{\partial u}{\partial r}\right)^3+2\frac{u}{r^2}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+w\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+u\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{uw}{r^2}\frac{\partial u}{\partial z}+u\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}\\
&+\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z\partial t}+w\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+4\left(\frac{\partial w}{\partial z}\right)^3+3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z}+2u\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+w\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}\\
&+2u\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+w\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}+3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+2\frac{u^2}{r^2}\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial t}\\
&+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial t}+\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial t}+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2u\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+2w\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+2w\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z\partial t}.
\end{aligned}\right]\\
&+\alpha_2\left[4\frac{u^3}{r^3}+3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+4\left(\frac{\partial u}{\partial r}\right)^3+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}+4\left(\frac{\partial w}{\partial z}\right)^3+3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z}\right]\\
&+\beta_3\left[\begin{aligned}
&2\left(\frac{\partial w}{\partial r}\right)^4+8\frac{u^4}{r^4}+8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial z}\right)^4+16\frac{u^2}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+8\left(\frac{\partial u}{\partial r}\right)^4+16\frac{u^2}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial u}{\partial z}\right)^3\frac{\partial w}{\partial r}+8\frac{u^2}{r^2}\left(\frac{\partial w}{\partial r}\right)^2\\
&+8\frac{u^2}{r^2}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^3+12\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial r}\right)^2+16\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial r}\right)^2+8\left(\frac{\partial w}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+16\frac{u^2}{r^2}\left(\frac{\partial w}{\partial z}\right)^2\\
&+16\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+8\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+16\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial w}{\partial z}\right)^4.
\end{aligned}\right]\\
&+\frac{16\sigma^* T_h^{\,3}}{3k_1^{\,*}}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{\sigma B_0^{\,2}u^2}{1-ct}
\end{aligned}
\tag{A13}
$$

$$
\begin{aligned}
&\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+\\
&\alpha_1\left[\begin{array}{l}
-2\frac{u^2}{r^3}-\frac{2}{r^2}\frac{\partial u}{\partial t}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+5\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{w}{r}\frac{\partial^2 u}{\partial r\partial z}+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2\\
-2\frac{u}{r^2}\frac{\partial u}{\partial r}+4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 w}{\partial r\partial z^2}+3\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}\\
-2\frac{w}{r^2}\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^2\partial t}+w\frac{\partial^3 u}{\partial z^3}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2w\frac{\partial^3 u}{\partial r^2\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
+10\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial^2 u}{\partial r\partial t}+\frac{\partial^3 w}{\partial r\partial z\partial t}+u\frac{\partial^3 u}{\partial r^2\partial z}+2\frac{\partial^3 u}{\partial r^2\partial t}+2u\frac{\partial^3 u}{\partial r^3}+2\frac{u}{r}\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}
\end{array}\right]\\
&+\alpha_2\left[\begin{array}{l}
-4\frac{u^2}{r^3}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\\
+2\frac{\partial^2 w}{\partial r\partial z}\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}
\end{array}\right]\\
&+\beta_3\left[\begin{array}{l}
-8\frac{u^3}{r^4}+8\frac{u^2}{r^2}\frac{\partial^2 u}{\partial r^2}+\frac{8}{r}\left(\frac{\partial u}{\partial r}\right)^3+24\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\\
+16\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+8\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}-4\frac{u}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+16\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
+4\frac{u^2}{r^2}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial^2 u}{\partial r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}-8\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}\\
+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}+4\frac{u^2}{r^2}\frac{\partial^2 u}{\partial z^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+4\frac{u}{r^2}\left(\frac{\partial u}{\partial z}\right)^2\\
+4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}-8\frac{u}{r^2}\left(\frac{\partial w}{\partial z}\right)^2+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}\\
-8\frac{u^2}{r^3}\frac{\partial u}{\partial r}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}
\end{array}\right]-\frac{\sigma B_0^2 u}{1-ct}-\frac{\mu_{nf}}{K}u
\end{aligned}
\tag{A10}
$$