FAZ Publishing
JCF
Journal of Complex Flow, Vol. 3 No. 2 (2021) p. 38-41

Journal of Complex Flow

Journal homepage : www.fazpublishing.com/jcf

e-ISSN : 2672-7374

Exact Solutions to the Navier–Stokes Equations for Nonlinear Viscous Flows by Undetermined Coefficient Approach

Muhammad Haizat Harun[1], Muhammed Abdelfattah Sayed Abdelaal[2], Djamal Hissein Didane[3], Izzuddin Zaman[3], Bukhari Manshoor[3,*]

[1] Barry Callebout (M) Sdn Bhd, Jalan Sultan Mohamed 1, Port Klang, Selangor, MALAYSIA

[2] Engineers House Academy, Manshiet El Bakry Street, Qesm Heliopolis, EGYPT

[3] Faculty of Mechanical and Manufacturing Engineering,
Universiti Tun Hussein Onn Malaysia, Batu Pahat, 86400, MALAYSIA

*Corresponding Author

Received 3 August 2021;
Accepted 26 September 2021;
Available online 30 Oct. 2021

Abstract: Exact solutions to the Navier–Stokes's equations for nonlinear viscous flows are investigated. These solutions belong to Lin's class of solutions, which are velocities that are linearly dependent on a portion of coordinates. This enables these solutions to be used to understanding large-scale processes of nature for example in the ocean and atmospheric phenomena. The precise solution obtained describes the flow of a vertical vortex fluid. The consideration of inertia forces and the nonuniform velocity distribution on the fluid layer's free boundary results in a vertical twist in the fluid. This solution describes counterflows of an incompressible fluid for flows in a thin layer. As a result, the exact solution of Navier–Stokes's equations for nonlinear viscous flows are obtained. This solution describes counterflows of an incompressible fluid for thin-layer flows. As a result, the exact solution of Navier-Stokes's equations obtained describes a novel mechanism of momentum transfer in a fluid. The objective of this research is to obtain the solutions of the exact solution of Navier-Stokes's equations and analyze the results. From the finding from the analyzed results, it is demonstrated that stagnation points exist for the flow of a vertical vortex fluid in an infinite layer with permeable boundaries.

Keywords: Couette flows, exact solutions, nonlinear system, Navier-Stokes's equation

1. Introduction

The Couette flow and Stokes flows of two types are examples of nonlinear flows of a viscous incompressible fluid [1, 2]. Corresponding precise solutions are frequently thought of as linear Stokes's system solutions [3, 4], but they are also complete Navier–Stokes's system solutions. The study of nonlinear of a viscous incompressible fluid flows can be reduced in some situations to solving issues of fluid motion in no inertial coordinate systems, such as the Ekman and Poiseuille flows [2, 5]. Shear flows are an appropriate notion for understanding large-scale processes for example in the ocean and atmospheric phenomena. Some researchers provide a collection of known exact solutions for layered and shear flows. The beginning of the systematic study of nonlinear of a viscous incompressible fluid flows was laid in [6], in which a few corresponding exact solutions of the Navier–Stokes's equations were given. The Navier–Stokes's system became overdetermined, which necessitated the study of such flows. Conditions for its solvability were determined in by Yu. D. Shmyglevskii, along with a classification of exact solutions [7]. Aristov et. al. investigated the potential of obtaining accurate solutions for three-dimensional Couette flows [8]. The velocity components were linearly dependent on two coordinates in the solutions that resulted as in Note on a Class of Exact Solutions in Magneto-Hydrodynamics, by Lin [9].

The solvability of an overdetermined system of Navier–Stokes's equations for three-dimensional nonlinear of a viscous incompressible fluid flows of a viscous incompressible fluid is discussed in this study. The velocity components of the equations of motion can be reduced to two connected quasilinear parabolic equations using a consistent analysis of their compatibility. The reduced equations allow the development of numerous classes of exact nontrivial solutions, with the solutions [10] that have the Lin class structure stated as a particular case.

**Corresponding author: bukhari@uthm.edu.my*

2. Mathematical Model – Undetermined Approach

We write the Navier-Stokes equation for viscous fluid steady motion and the incompressibility equation, both projected on the axes of a rectangular Cartesian coordinate system. We obtain the stationary system of nonlinear partial differential equations shown below:

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} = -\frac{\partial P}{\partial x} + v\left(\frac{\partial V_x^2}{\partial x^2} + \frac{\partial V_x^2}{\partial y^2} + \frac{\partial V_x^2}{\partial z^2}\right) \quad (1)$$

$$V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} = -\frac{\partial P}{\partial y} + v\left(\frac{\partial V_y^2}{\partial x^2} + \frac{\partial V_y^2}{\partial y^2} + \frac{\partial V_y^2}{\partial z^2}\right) \quad (2)$$

$$V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} = -\frac{\partial P}{\partial z} + v\left(\frac{\partial V_z^2}{\partial x^2} + \frac{\partial V_z^2}{\partial y^2} + \frac{\partial V_z^2}{\partial z^2}\right) \quad (3)$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \quad (4)$$

Here, V_x, V_y and V_z are velocity vector components; P is pressure divided by density ρ, v is the coefficient of kinematic viscosity. The exact solution of the system (1) is found in the following form:

$$\begin{aligned} V_x(x,y,z) &= U(z) \\ V_y(x,y,z) &= V(z) + x v_1(z) \\ V_z &= w \end{aligned} \quad (5)$$

The exact solutions (5) describe the flow of a permeable vertical vortex fluid. $V_z = w = \text{const.} > 0$ is assumed to be a constant value for the vertical velocity component. A fluid outlet from the fluid layer boundaries is defined by the velocity V_z. The fluid's motion is assumed to be isobaric, which means that the pressure is a constant function.

$$P(x,y,z) = P_o \quad (6)$$

The exact solutions (5) generalize the solution family for isobaric [11], gradient [12] and convective [13] large-scale flows of a viscous incompressible fluid. We substitute the solutions (5) and (6) in the system (1-4) and project the obtained expressions on the axes O_x, O_y, O_z. We obtain the system of partial differential equations,

$$\begin{aligned} w\frac{\partial U}{\partial z} &= v\frac{\partial^2 U}{\partial z^2} \\ Uv_1 + w\left(\frac{\partial V}{\partial z} + \frac{\partial v_1}{\partial z}x\right) &= v\left(\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 v_1}{\partial z^2}x\right) \end{aligned} \quad (7)$$

Because all the functions U, V, and v_1 are dependent exclusively on the coordinate z, the partial derivatives in equations (7) can be denoted with a prime. For the definition of the functions U, V, and v_1, we use the undetermined coefficients approach and create the following system of ordinary differential equations:

$$\begin{aligned} &vU'' - wU' = 0; \\ &vV'' - wV' - Uv_1 = 0 \\ &vv_1'' - wv_1' = 0 \end{aligned} \quad (8)$$

The resultant system of equations is reduced to a nondimensional form. To accomplish so, we'll use the scale variables listed below. A scale l characterises the horizontal coordinate x, while the fluid layer thickness h characterises the vertical coordinate. Similarly, two velocity scales are introduced. In this situation, the homogeneous terms of the hydrodynamic velocity fields have the following scale variables.

$$[U], w$$

The second equation of the system (8) transformed by considering the chosen characteristic dimensions of velocities and coordinates as,

$$vV''\frac{[U]}{h^2} - V'\frac{w[U]}{h} - Uv_1[U]\frac{[U]}{l} = 0;$$

$$vV''\frac{1}{h^2} - V'\frac{w}{h} - Uv_1\frac{[U]}{l} = 0$$

Multiplying the transformed equation by the number h_2/v. We obtain,

$$V'' - wV'\frac{[w]h}{v} - Uv_1\frac{[U]h^2}{vl} = 0;$$

$$V'' - \mathrm{Re}_w wV' - \delta^2 \mathrm{Re}_u Uv_1 = 0.$$

The Reynolds number relative to the horizontal (longitudinal) velocity is $Re_u=[U]l/v$; the Reynolds number relative to the vertical (transverse) velocity is $Re_w=wh/v$; and the characteristic scale ratio is $\delta=h/l$. In the same way, the remaining two equations are reduced to a nondimensional form. The equations that arise are as follows:

$$\begin{aligned} &U'' - \mathrm{Re}_w U' = 0; \\ &V'' - \mathrm{Re}_w V' - \delta^2 \mathrm{Re}_u Uv_1 = 0; \\ &v_1'' - \mathrm{Re}_w v_1' = 0. \end{aligned} \quad (9)$$

3. Exact Solutions for the Nonlinear System of Equations

The general solution for the system of ordinary differential equations (9) has the form,

$$U = \frac{\exp(\mathrm{Re}_w z)C_1}{\mathrm{Re}_w} + C_2 \quad (10)$$

$$\begin{aligned} V = &\frac{\mathrm{Re}_u \mathrm{Re}_w{}^2 \delta^2 C_2 C_6}{\mathrm{Re}_w} z \\ &+ \frac{\exp(\mathrm{Re}_w z)\mathrm{Re}_u \delta^2 C_1 C_5}{2\mathrm{Re}_w{}^4} \\ &+ \frac{\exp(\mathrm{Re}_w z)C_3}{\mathrm{Re}_w} \\ &+ \frac{\exp(\mathrm{Re}_w z)\mathrm{Re}_u \delta^2 (C_1C_6 + C_2C_6)}{\mathrm{Re}_w{}^2}\left(z - \frac{1}{\mathrm{Re}_w}\right) \\ &+ C_4 \end{aligned} \quad (11)$$

where C_i, i=1,6 are integration constants.

Published by FAZ Publishing
http://www.fazpublishing.com/jcf

To find the integration constants, we use the boundary value issue. On the bottom boundary, the no-slip criterion is met. The velocities at the top permeable border are represented in non-dimensional form as,

$$\text{for } z=0 \quad U(0)=V(0)=v_1(0)=0;$$

$$\text{for } z=1 \; U(1)=\cos\alpha,\; V(1)=\sin\alpha,\; v_1(1)=\frac{Ta}{2\,\mathrm{Re}_u} \qquad (12)$$

The modified Taylor number is $Ta=2\Omega l^2/\nu$, and Ω represents the value of the vertical vorticity component on the fluid layer's upper boundary. The solution (11) of the equation system (10) then has the form, due to the boundary conditions (12).

$$U=\frac{1-\exp(\mathrm{Re}_w z)}{1-\exp(\mathrm{Re}_w)}\cos\alpha \qquad (13)$$

$$\begin{aligned} V=&\frac{1-\exp(\mathrm{Re}_w z)}{1-\exp(\mathrm{Re}_w)}\sin\alpha-\frac{Ta\delta^2\cos\alpha}{4\left[1-\exp(\mathrm{Re}_w)\right]^3\mathrm{Re}_w^2}\\ &\{\exp(2\,\mathrm{Re}_w z)\left[-1+\exp(\mathrm{Re}_w)\right]\\ &+\exp(2\,\mathrm{Re}_w)\left[1+\exp(\mathrm{Re}_w z)\right]\\ &+(1-z)\left(4\exp(\mathrm{Re}_w+\mathrm{Re}_w z)\mathrm{Re}_w-2\,\mathrm{Re}_w\right)\\ &-\exp(\mathrm{Re}_w)\left(1+2\,\mathrm{Re}_w(2+z)\right)\\ &-\exp(\mathrm{Re}_w z)\left(1+2\,\mathrm{Re}_w(1+2z)\right)\} \end{aligned} \qquad (14)$$

$$v_1=\frac{Ta}{2\,\mathrm{Re}_u}\frac{1-\exp(\mathrm{Re}_w z)}{1-\exp(\mathrm{Re}_w)} \qquad (15)$$

We investigate the solution for equations (13), (14) and (15). The functions U and v_1 are monotonic, and at the lower boundary, $z=0$, they become zero. Figures 1 and 2 illustrate the graphs of the functions $U(z)$ and $v_1(z)$, respectively.

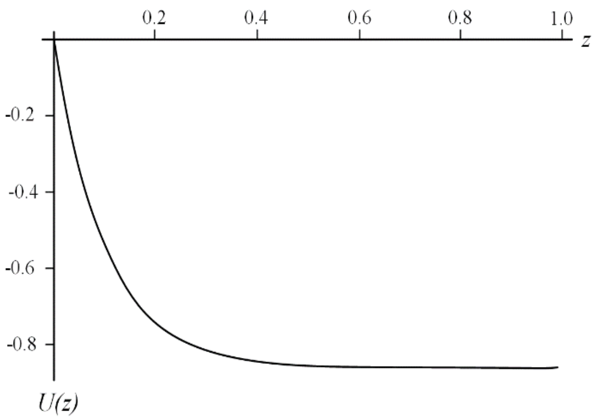

Fig. 1 - Schematic diagram of circular pipe

Let us now investigate the velocity V. We represent the function V in a multiplicative form as,

$$V(z)=z.f(z)$$

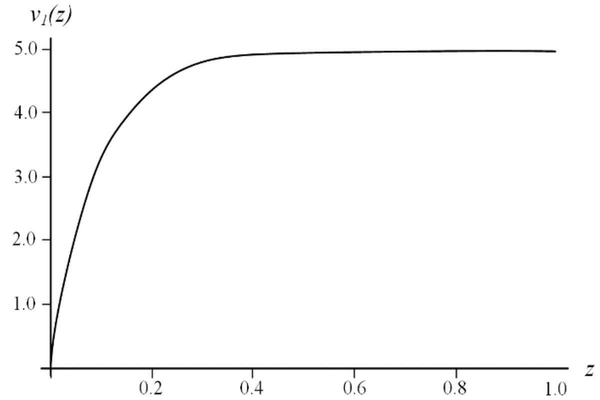

Fig. 2 - Schematic diagram of circular pipe

We investigate the velocity $V(z)$ for the possibility of the existence of counterflows, *i.e.* the existence of zeros on the interval, $z\epsilon(0;1)$. The condition of the existence of such zeros is as follows:

$$f(0).f(1)<0$$

The values of the function $f(z)$ at the point $f=0$ and $z=0$ is calculated by the formulas.

$$\begin{aligned} f(0)=\lim_{z\to 0}\frac{V(z)}{z}=&-\frac{\mathrm{Re}_w\sin\alpha}{1-\exp(\mathrm{Re}_w)}\\ &+\frac{\exp(2\,\mathrm{Re}_w)Ta\delta^2\cos\alpha}{4\left(1-\exp(\mathrm{Re}_w)\right)^3\mathrm{Re}_w}\\ &+\frac{4\exp(\mathrm{Re}_w)(1-\mathrm{Re}_w)Ta\delta^2\cos\alpha}{4\left(1-\exp(\mathrm{Re}_w)\right)^3\mathrm{Re}_w}\\ &-\frac{(2\,\mathrm{Re}_w+5)Ta\delta^2\cos\alpha}{4\left(1-\exp(\mathrm{Re}_w)\right)^3\mathrm{Re}_w} \end{aligned}$$

$$f(1)=\sin\alpha$$

On the *z*-axis, the function $V(z)$ has one zero point (0;1). As a result, the counterflow in the current reaches a point of stagnation, resulting in the presence of counterflows in the fluid, as seen in Fig. 3.

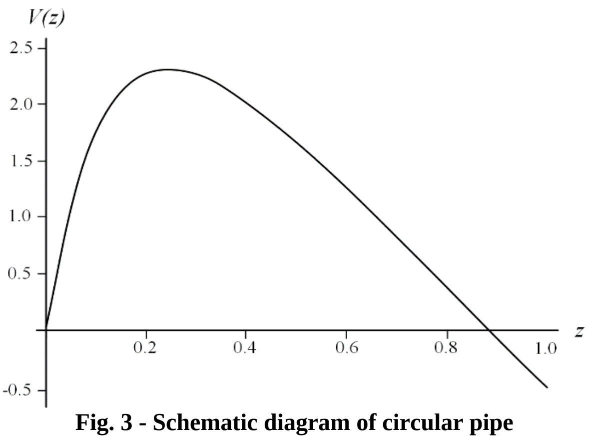

Fig. 3 - Schematic diagram of circular pipe

Published by FAZ Publishing
http://www.fazpublishing.com/jcf

For various values of the modified Taylor number, Fig. 4 shows the localization of the roots of the function *V(z)* on the interval zϵ (0;1). When the centrifugal forces exceed the viscous friction forces, the solution *V(z)* is localised around the upper boundary, according to the physical meaning of the Taylor number.

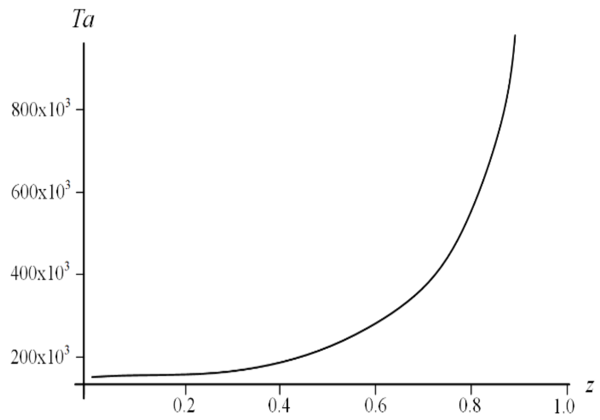

Fig. 4 - Schematic diagram of circular pipe

4. Conclusion

For three-dimensional Navier–Stokes's equations for nonlinear viscous flows, a generalisation of the stable classical Couette solution was found. It has been demonstrated that stagnation points can exist in hydrodynamic fields when specific physical constants and boundary conditions are enforced. The localization of the roots of the exact solutions was used together with undetermined coefficient method to investigate the qualitative and quantitative features of the obtained exact solutions in the class of linearly increasing velocities along horizontal coordinates.

Acknowledgement

The author would also like to thank the Faculty of Mechanical and Manufacturing Engineering, Universiti Tun Hussein Onn Malaysia, and Centre for Energy and Industrial Environment Studies (CEIES), for the supports. Communication of this research is made possible through Barry Callebout (M) Sdn Bhd and Engineers House Academy for supporting data and technical advice.

References

[1] Drazin, P.G., and Riley, N., "The Navier–Stokes Equations: A Classification of Flows and Exact Solutions." (Cambridge Univ. Press, Cambridge, 2006).

[2] Aristov, S.N., Knyazev, D.V., and Polyanin, A.D., "Exact Solutions of the Navier–Stokes Equations with the Linear Dependence of Velocity Components on Two Space Variables," *Theor. Found. Chem. Eng.* 43(5) (2009): 547–566

[3] Pukhnachev, V.V., "Symmetries in Navier–Stokes Equations," *Usp. Mekh*. 4(1) (2006): 6–76

[4] Prosviryakov, E.Y., "Waves of Pressure in Viscous Incompressible Fluid," in *Mechanics, Resource and Diagnostics of Materials and Structures, MRDMS 2017*, AIP Conference Proceedings.

[5] Ludlow, D.K., Clarkson, P.A., and Bassom, A.P., "Nonclassical Symmetry Reductions of the Two-Dimensional Incompressible Navier-Stokes Equations," *Stud. Appl. Math.*, 103 (1999): 183–240.

[6] Berker, R., "Sur Quelques cas d'int´egration des Equations du Mouvement d'un Fluide Visqueux Incompressible." (Taffin-Lefort, Paris–Lille, 1936).

[7] Shmyglevskii, Y.D., "On Isobaric Planar Flows of a Viscous Incompressible Liquid," *USSR. Comput. Math. Math. Phys.* 25(12) (1985): 1895–1898 (1985)

[8] Aristov, S.N., and Prosviryakov, E.Y., "Inhomogeneous Couette Flow," *Nelin. Dyn.* 10(2) (2004): 177–182

[9] Lin, C.C., "Note on a Class of Exact Solutions in Magneto-Hydrodynamics," *Arch. Rational Mech. Anal.* 1 (1958): 391–395

[10] Privalova, V.V. and Prosviryakov, E.Y., "An Inhomogeneous Couette-type Flow with a Perfect Slip Condition at the Lower Boundary of an Infinite Fluid Layer." AIP Conference Proceedings. 2176.

[11] Aristov, S.N., Privalova, V.V., and Prosviryakov, E.Y., "Stationary Nonisothermal Couette Flow - Quadratic Heating of the Upper Boundary of the Fluid Layer." *Nelin. Dinam.*, 12(2) (2016): 167 – 178.

[12] Ambrozhevich, A.V., Migalin, K.V., and Sereda, V.A., "Optimization of Dynamic Performance for a Ground Catapult with Reusable Supply of the Working Fluid." *Russian Aeronautics*, 59(4) (2016): 441 - 446

[13] Aristov, S.N., and Prosviryakov, E.Y., "Large-scale Flows of Viscous Incompressible Vortical Fluid." Russian Aeronautics. 58(4) (2015): 413-418