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Natural Heat Convection through Vertical Flat Plates by Volume of Parameter Method

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Abstract: Natural heat convection for the flow of a third-grade fluid through two parallel vertical infinite flat plates is investigated. As the nonlinear differential equation which governs the flow model can be obtained by using the conversation of laws, the Variation of Parameters Method (VPM) is then applied to solve the aforementioned differential equation. The VPM is used to determine the semi-exact solution to the problem. The proposed VPM is applied without any discretization, perturbation, transformation or restrictive assumptions and is free from round off errors and calculation. A numerical solution is also sought using Runge-Kutta method. The results show that the Runge-Kutta numerical solution and the VPM has an excellent agreement. Effects of flow parameters *β,* Eckert number, *Ec* and Prandtl number, *Pr* on the velocity profile and temperature distribution were demonstrated graphically with comprehensive discussions. Numerical results reveal the complete reliability of the proposed VPM.

Keywords: Natural heat convection, Variation of Parameters Method (VPM), vertical parallel flat plates, third-grade fluid, non-Newtonian fluids.

1. Introduction

The importance of non-Newtonian fluids has become prominent with developments in industries like oil and gas, polymer, pulp *etc* [1]. Most of the industries mentioned basically plays a significant role in overall behaviour of the flow especially natural heat convection [2-4]. This is due to the most of solutions for melts of polymers, soap, biological solutions, paints, tars, asphalts and glues involved with the non-Newtonian fluids [5]. As we know that the non-Newtonian fluids is a complex nature, hence, to describe the characteristics of all non-Newtonian fluids in mathematical model is very difficult. Nowadays, there are many researchers were introduced a mathematical model to discuss flow of non-Newtonian flows. For the discussion on non-Newtonian flows, third-grade fluids mostly choose by the researcher to develop a mathematical model due to the simpler mathematical simulation [6-8].

Despite the great achievement in modelling flow models for non-Newtonian flows, especially that related to the heat convection, still there is a lot of space for new ideas and methodologies. For the flow of a specific class of such as third-grade fluids, natural heat convection that occurred through parallel vertical flat plates was been discussed by R.W. Bruce et. al [9]. Other researchers presented a comprehensive thermodynamic analysis of constitutive relations for some classes of non-Newtonian fluids [10]. Besides these two classical references that used the third-grade fluids for study the natural convection through the parallel plates, there are many other researchers have used this constitutive relation to model the flow of non-Newtonian fluids [11-13].

Most of the physical problems involving non-Newtonian fluids may not possible to have an exact due to the nonlinearities. Therefore, over the year there are different analytical techniques have been developed. One

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of such techniques is Variation of Parameters Method (VPM) which is very effective and easy to apply [14-16], have applied it to find the solution for many abstract problems.

Motivated by the ongoing research in this direction, the objective of this study is to apply VPM to study the natural heat convection through vertical flat plates. Besides to analytically compute the natural heat convection by VPM, the solution will be compared with the numerical solution obtained by using Runge-Kutta method in order to verify the capability of the VPM method. The behaviour of flow for ranges of physical parameters is also will discussed in detail.

2. Governing Equations

The governing equations for natural heat convection through vertical flat plates of third grade fluid given by a basic law of conservation of mass, momentum and energy equations which given by,

$$\nabla .V = 0 \tag{1}$$

$$\rho \frac{DV}{Dt} = -\nabla p + div\,\tau \tag{2}$$

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \tau .L \tag{3}$$

where V is the velocity field, ρ and p are the constant density and pressure of fluid, τ is a stress tensor, k is the thermal conductivity, while c_p is the specific heat respectively. For the term of energy equation, L denotes a gradient of velocity field and for the material derivative *D/Dt*, it can be defined as,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(V.\nabla\right)$$

The third-grade fluid stress sensor are given by the following,

$$\tau = -pI + S_1 + S_2 + S_3 \tag{4}$$

where,

$$S_1 = \mu A_1$$

$$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2$$

$$S_3 = \beta_1 A_3 + \beta_2 \left(A_1 A_2 + A_2 A_1\right) + \beta_3 \left(tr\,A_2\right) A_1$$

For the stress sensor of the third-grade fluid, μ is the dynamics viscosity while the α_1, α_2, β_1, β_2 and β_3 are material constant. A_1, A_2 and A_3 representing the Rivilin-Ericksen tensor and define by the following equation, where, '+' represent the transpose of a matrix.

$$A_1 = L + L^+ \tag{5}$$

$$A_n = \frac{dA_{n-1}}{dt} + A_{n-1}\left(\nabla .V\right) + A_{n-1}\left(\nabla .V\right)^+ , \quad n \ge 1 \tag{6}$$

3. Problem Formulation

For this study, an incompressible steady flow of a third grade between two parallel flat plates was considered. The plates are fixed vertically and are distant *2h* apart, as shown in Fig. 1. The temperature different at both walls will results an increase in fluid temperature near the wall on the left side (x=-h) while falls in the fluid temperature near the right wall (x=+h).

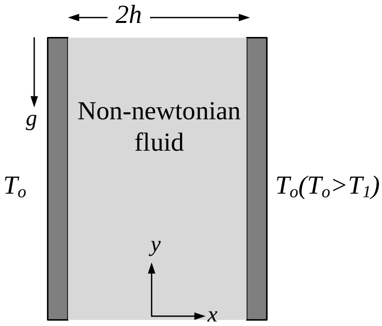

Fig. 1 – Physical model of problem

The one-dimensional steady velocity field can be written in the form,

$$V = \left[0, v(x), 0\right],\; T = T(x) \tag{7}$$

Continuity equation (1) is satisfied identically. By absence of body forces, momentum and energy equations (2) and (3) hence become,

$$\mu \frac{d^2 v}{dx^2} + 6\left(\beta_3\right)\left(\frac{dv}{dx}\right)^2 \frac{d^2 v}{dx^2} + \rho_o \gamma \left(T - T_m\right) g = 0 \tag{8}$$

$$k\frac{d^2 T}{dx^2} + \mu \left(\frac{dv}{dx}\right)^2 + 2\left(\beta_3\right)\left(\frac{dv}{dx}\right)^4 = 0 \tag{9}$$

subject to the following boundary conditions.

$$v(-h) = 0, \qquad v(h) = 0$$

$$T(-h) = T_o, \qquad T(h) = T_1$$

Suitable dimensionless variables can be introduced so that the analysis only consider the dimensionless parameter,

$$v^* = \frac{v}{V_o},\quad x^* = \frac{x}{h},\quad T^* = \frac{T - T_m}{T_o - T_1} \tag{10}$$

and dropping asterisk for simplicity, dimensionless form of the momentum and energy equations, (8) and (9) with the boundary conditions becomes,

$$\frac{d^2 v}{dx^2} + 6\beta \left(\frac{dv}{dx}\right)^2 \frac{d^2 v}{dx^2} + T = 0 \tag{11}$$

$$\frac{d^2 T}{dx^2} + Ec.\Pr \left(\left(\frac{dv}{dx}\right)^2 + 2\beta \left(\frac{dv}{dx}\right)^4\right) = 0 \tag{12}$$

with following boundary conditions,

$$v(-1) = 0, \qquad v(1) = 0$$

$$T(-1) = \frac{1}{2}, \qquad T(1) = -\frac{1}{2}$$

where, Eckert number, $Ec = \dfrac{V_o^{\,2}}{c_p\left(T_1 - T_o\right)}$, Prandtl number $\Pr = \dfrac{\mu c_p}{k}$ and viscoelastic parameter, $\beta = \left(\dfrac{\beta_3 V_o^{\,2}}{\mu h^2}\right)$.

4. Variation of Parameter Method (VPM)

To illustrate the basic concept of the variation of parameter method for differential equations, let consider the general differential equation in operator form,

$$Lv(x) + Nv(x) + R(v) = g(x) \tag{13}$$

where L is a higher order linear operator, R is a linear operator of order less then L, N is a nonlinear operator, and g is a source term. From here, we have following general solution of equation.

$$\begin{aligned} v(x) &= \sum_{i=0}^{n-1} \frac{A_i x^i}{i!} \\ &+ \int_0^x \lambda(x,s)\left(-Nv(s) - Rv(s) + g(s)\right) ds \end{aligned} \tag{14}$$

where n is the order of given differential equation and A_i's are unknowns which can be further determined by initial/boundary conditions. Here, $\lambda(y,s)$ is a multiplier which can be obtained with the help of Wronskian technique as,

$$\lambda(x,s) = x - s$$

From here, we can have the following iterative scheme.

$$\begin{aligned} v_{k+1}(x) &= \sum_{i=0}^{n-1} \frac{A_i x^i}{i!} \\ &+ \int_0^x (x-s)\left(-Nv(s) - Rv(s) + g(s)\right) ds, \end{aligned} \tag{15}$$

where $k = 0,1,2,\ldots$

5. Solution of the Problem

By using the equations (11) and (12) with the boundary conditions in equation (15), we have the following iterative scheme.

For velocity;

$$\begin{aligned} v_{k+1}(x) &= A + Bx \\ &- \int_0^x (x-s)\left(6\beta\left(\frac{dv_n(s)}{ds}\right)^2 \frac{d^2 v_n(s)}{ds^2} + T_n(s)\right) ds, \end{aligned} \tag{16}$$

where,

$$A = v(0), \quad B = \frac{dv}{dx}(0)$$

For temperature,

$$\begin{aligned} T_{k+1}(x) &= C + Dx \\ &- \int_0^x (x-s)\left(Ec.\Pr\left(\left(\frac{dv_n(s)}{ds}\right)^2 + 2\beta\left(\frac{d^2 v_n(s)}{ds^2}\right)^4\right)\right) ds, \end{aligned} \tag{17}$$

where,

$$C = T(0), \quad D = \frac{dT}{dx}(0)$$

can be determined by using the boundary conditions. Few iterations of the solution are given below.

$$v_1 = A + Bx - \frac{1}{2}Cx^2 - \frac{1}{6}Dx^3$$

$$\begin{aligned} v_2 &= A + Bx + \left(-\frac{1}{2}C + 3\beta B^2 C\right)x^2 + \left(-2\beta BC^2 + \beta B^2 D - \frac{1}{6}D\right)x^3 \\ &+ \left(-\frac{3}{2}\beta BCD + \frac{1}{12}Ec.\Pr\beta B^4 + \frac{1}{24}Ec.\Pr B^2 + \frac{1}{2}\beta C^3\right)x^4 \\ &+ \left(\frac{3}{5}\beta C^2 D - \frac{3}{10}\beta D^2 B\right)x^5 + \left(\frac{1}{4}\beta CD^2\right)x^6 + \left(\frac{1}{28}\beta D^3\right)y^7 \end{aligned}$$

$$T_1 = C + Dx + Ec.\Pr\left(-\frac{1}{2}B^2 - \beta B^4\right)x^2$$

$$\begin{aligned} T_2 &= C + Dx + Ec.\Pr\Bigg[\left(-\frac{1}{2}C + 3\beta B^2 C\right)x^2 \\ &+ \left(\frac{1}{3}BC + \frac{4}{3}\beta B^2 C\right)x^3 \\ &+ \left(\frac{1}{12}BD - \frac{1}{12}C^2 + \frac{1}{3}\beta B^3 D - \beta B^2 C^2\right)x^4 \\ &+ \left(-\frac{1}{20}CD + \frac{2}{5}\beta BC^3 - \frac{3}{5}\beta B^2 D^2\right)x^5 \\ &+ \left(-\frac{1}{15}\beta C^4 - \frac{1}{120}D^2 - \frac{1}{10}\beta B^2 D^2 + \frac{2}{5}\beta BC^2 D\right)x^6 \\ &+ \left(-\frac{2}{21}\beta C^3 D + \frac{1}{7}\beta BCD^2 + \frac{2}{5}\right)x^7 \\ &+ \left(-\frac{3}{56}\beta C^2 D^2 + \frac{1}{56}\beta D^3 B\right)x^8 \\ &+ \left(-\frac{1}{72}\beta CD^3\right)x^9 + \left(-\frac{1}{720}\beta D^4\right)x^{10}\Bigg] \end{aligned}$$

6. Results and Discussion

Results for the non-Newtonian third grade flow down an inclined plane by using VPM will discussed. Fig. 1 shows the variations in velocity profile for different values of non-Newtonian parameter β and fixed the value of m. It can be observed that as the value of β increases, velocity decreases. For β = 0 solution reduces to the case of Newtonian fluid, which is represented by solid line in the graph. From Fig. 2 it can be observed that for fixed value of β velocity increases by increasing m.

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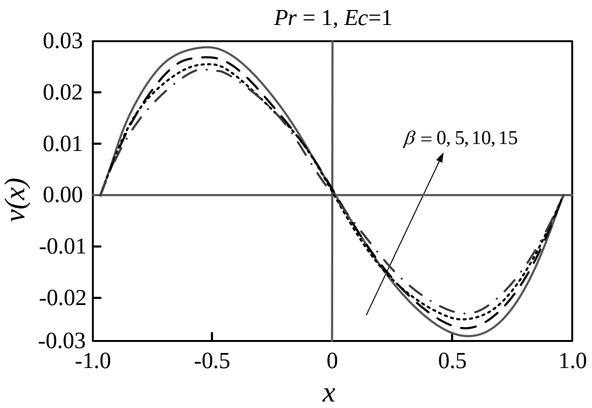

Fig. 2 – Effect of β on velocity profile

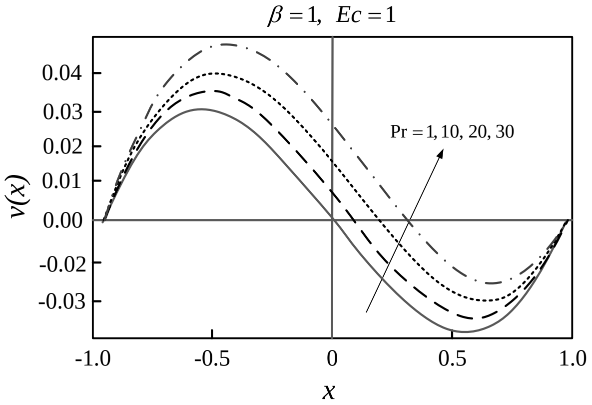

Fig. 3 – Effects of Prandtl number on velocity profile

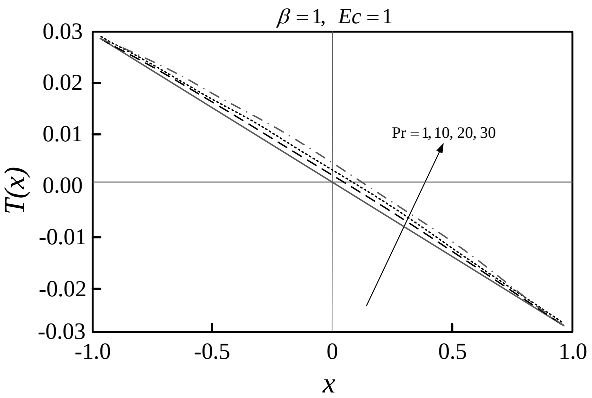

Fig. 4 – Effects of Prandtl number on temperature distribution

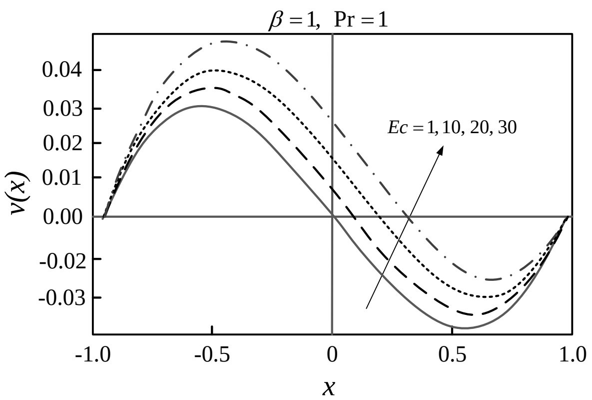

Fig. 5 – Effects of Eckert number on velocity profile

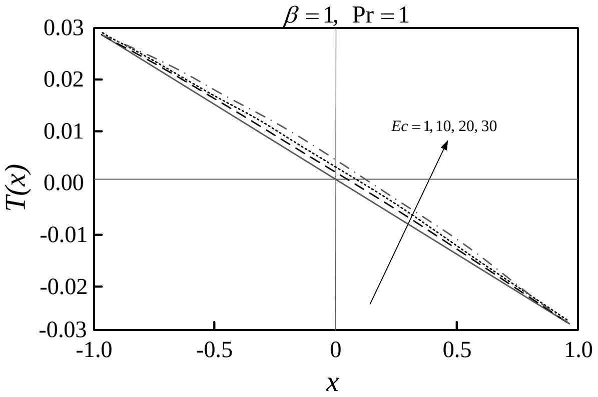

Fig. 6 – Effects of Eckert number on temperature distribution

The problem of natural heat convection through vertical flat plates also was numerically solved by using Runge-Kutta method for the case of $\beta = 0.5$, Eckert number, $Ec = 1$ and Prandtl number, $\Pr = 1$ in order to support the analytical solution by VPM method. Comparison for both results for velocity profile and temperature distribution had shown an excellent agreement, which approve the effectiveness of VPM. The comparison results clearly present in Table 1 for the velocity profile while Table 2 shows the results for temperature distribution.

Table 1 – Comparison between VPM and numerical solutions for velocity profile, *v(x)*

x	VPM	Numerical	Error
-1	0	0	0
-0.9	0.015226378	0.015226372	3.94E-05
-0.8	0.033923604	0.033923618	4.13E-05
-0.7	0.038975352	0.038975343	2.31E-05
-0.6	0.032183544	0.032183534	3.11E-05
-0.5	0.032369454	0.032369437	5.25E-05
-0.4	0.027143138	0.027143126	4.42E-05
-0.3	0.025231439	0.025231452	5.15E-05
-0.2	0.016274634	0.016274626	4.92E-05
-0.1	0.003874218	0.003874215	7.74E-05
0.0	0.000922405	0.000922405	0.00E+00
0.1	-0.006937355	-0.006937351	5.77E-05
0.2	-0.015143973	-0.015143976	1.98E-05
0.3	-0.021839825	-0.021839821	1.83E-05
0.4	-0.028257613	-0.028257621	2.83E-05
0.5	-0.030406214	-0.030406223	2.96E-05
0.6	-0.031223835	-0.031223841	1.92E-05
0.7	-0.029073293	-0.029073287	2.06E-05
0.8	-0.023274354	-0.023274348	2.58E-05
0.9	-0.013736403	-0.013736413	7.28E-05
1	0	0	0

Table 2 – Comparison between VPM and numerical solutions for temperature distribution, *T(x)*

x	VPM	Numerical	Error
-1	0.5	0.5	0
-0.9	0.450783529	0.450783535	1.33E-06
-0.8	0.400246306	0.400246312	1.50E-06
-0.7	0.378261537	0.378261529	2.11E-06
-0.6	0.309367078	0.309367086	2.59E-06
-0.5	0.252472403	0.252472415	4.75E-06
-0.4	0.201548965	0.201548973	3.97E-06
-0.3	0.151376653	0.151376648	3.30E-06
-0.2	0.101925345	0.101925352	6.87E-06
-0.1	0.052037254	0.052037249	9.61E-06
0	0.002174325	0.002174324	4.60E-05
0.1	-0.047895328	-0.047895324	8.35E-06
0.2	-0.098174536	-0.098174528	8.15E-06
0.3	-0.147634566	-0.147634571	3.12E-06
0.4	-0.198546725	-0.198546732	3.53E-06
0.5	-0.247634529	-0.247634535	2.42E-06
0.6	-0.298765434	-0.298765428	2.01E-06
0.7	-0.342543678	-0.342543683	1.46E-06
0.8	-0.400465233	-0.400465226	1.75E-06
0.9	-0.450098453	-0.450098447	1.33E-06
1	-0.5	-0.5	0

7. Conclusions

Natural heat convection of a non-Newtonian fluid between two vertically placed infinite parallel plates is considered. Natural heat convection for the flow is examined and Variation of Parameters Method is used to solve equations governing the flow. Effects of viscoelastic parameter β, Eckert number and Prandtl number on velocity profile and temperature distribution are recorded. Numerical solution is also sought and both solutions are compared which shows that there is an excellent agreement between both numerical and analytical solutions. Effects of involved physical parameters are also discussed graphically coupled with comprehensive discussions and explanations. From the obtained results, it can be concluded that VPM can successfully be applied to problems in natural heat convection, and generally in non-Newtonian fluid flow.

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